Math 245C Lecture 23 Notes

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1 Recovering Functions From Their Fourier Series

This lecture was given by a guest lecturer.

1.1 Recovering functions from their Fourier series

Theorem 1.1. *Suppose that $\Phi \in C(\mathbb{R}^n)$ satisfies $|\Phi(\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$, $|\Phi^\vee(x)| \leq C(1+|x|)^{-n-\varepsilon}$, and $\Phi(0) = 1$. Given $f \in L^1(\mathbb{T}^n)$, for any $t > 0$, set*

$$f^t(x) = \sum_{k \in \mathbb{Z}^n} \widehat{f}(k) \Phi(tz) e^{2\pi i k \cdot x}.$$

1. *If $f \in L^p(\mathbb{T}^n)$, then $\|f^t - f\|_p \to 0$ as $t \to 0$. If $f \in C(\mathbb{T}^n)$, then $f^t \to f$ uniformly as $t \to 0$.*
2. *$f^t(x) \to f(x)$ for every x in the Lebesgue set of f.*

Proof. First, let $\phi = \Phi^\vee$, and let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then $\widehat{\phi}_t(\xi) = \Phi(t\xi)$. Since $|\Phi(\xi)| \leq C(1+|\xi|)^{-n-\varepsilon}$, we have $\Phi \in L^1(\mathbb{R}^n)$. So $\phi \in C(\mathbb{R}^n)$. And, moreover,

$$\phi_t(x) = t^{-n}\phi(t^{-1}x) \leq Ct^{-n}(1+|t^{-1}x|)^{-n-\varepsilon} \leq Ct^{-n}(1+|x|)^{-n-\varepsilon},$$

where the last inequality holds for $t \ll 1$. Also,

$$\widehat{\phi}_t(\xi) = \Phi(t\xi) \leq C(1+t|\xi|)^{-n-\varepsilon} \overset{0<t<1}{\leq} C(t+t|\xi|)^{-n-\varepsilon} = Ct^{-n-\varepsilon}(1+|\xi|)^{-n-\varepsilon}.$$

Applying the Poisson summation formula for each fixed t, we get

$$\sum_{k \in \mathbb{Z}^n} \varphi_t(x-k) = \sum_{k \in \mathbb{Z}^n} \widehat{\phi}_t(k) e^{2\pi i k \cdot x} = \sum_{k \in \mathbb{Z}^n} \Phi(tk) e^{2\pi i k \cdot x} =: \psi_t(x) \in L^2(\mathbb{T}^n) \subseteq L^1(\mathbb{T}^n).$$

Then $\widehat{f * \psi_t}(k) = \widehat{f}(k)\widehat{\psi}_t(k)$, as $f, \psi_t \in L^1$ for each t. As $\psi_t \in L^2$, we have that $\psi_t(x) = \sum_{k \in \mathbb{Z}^n} \widehat{\psi}_t e^{2\pi k \cdot x}$, which means that $\widehat{\psi}_t(k) = \Phi(tk)$ (since the Fourier series coefficients agree). So

$$\widehat{f * \psi_t}(k) = \widehat{f}(k)\Phi(tk) = \widehat{f^t}(k).$$

So we get $f^t = f * \psi_t$ by taking the inverse Fourier transform. Hence, for all $1 \le p \le \infty$, by Young's inequality (and a theorem we have already proven),

$$\|f^t\|_p = \|f * \psi_t\|_p \le \|f\|_p \|\psi_t\|_1 \le \|f\|_p \|\phi_t\|_1 = \|f\|_p \|\phi\|_1.$$

So the operator $f \to f^t$ is uniformly bounded in L^p for $1 \le p \le \infty$.

Notice that Φ is continuous and $\Phi(0) = 1$. We have $f^t \to f$ uniformly if f is a trigonometric polynomial, i.e. $\widehat{f}(k) = 0$ for all but finitely many k: $f = \sum_{j=1}^m \widehat{f}(k_j) e^{2\pi i k_j \cdot x}$. By the Stone-Weierstrass theorem, the trigonometric polynomials are dense in $C(\mathbb{T}^n)$ and hence also dense in $L^p(\mathbb{T}^n)$. So for all $\varepsilon > 0$, there exists a trigonometric polynomial f_n such that $\|f - f_n\|_p \le \varepsilon$. Then

$$\begin{aligned}
\|f^t - f\|_p &\le \|f^t - f_n^t\| + \|f_n^t - f_n\|_p + \|f_n - f\|_p \\
&\le \|\phi\|_1 \|f - f_n\|_p + \|f_n^t - f_n\|_p + \|f_n - f\|_p \\
&\le (\|\phi\|_1 + 1)\varepsilon.
\end{aligned}$$

This proves the first statement.

For the second statement, without loss of generality, assume that 0 is a Lebesgue point of f. With $Q = [-1/2, 1/2)^n$, we have

$$f^t(0) = f * \psi_t(0) = \int_Q f(x)\psi_t(-x)\,dx = \int_Q f(x)\phi_t(-x)\,dx + \sum_{k\neq 0} \int_Q f(x)\phi_t(-x_k)\,dx$$

Since

$$|\phi_t(x)| \le Ct^{-n}(1 + t^{-1}|x|)^{-n-\varepsilon} = Ct^{\varepsilon}(t + |x|)^{-n-\varepsilon} \le Ct^{\varepsilon}|x|^{-n+\varepsilon},$$

we have

$$|\phi_t(x + k)| \le Ct^{\varepsilon}|-x + k|^{-n-\varepsilon} \le Ct^{\varepsilon}\left|\frac{k}{2}\right|^{-n-\varepsilon} = C2^{n+\varepsilon}t^{\varepsilon}|k|^{-n-\varepsilon}$$

for $k \neq 0$. So we get

$$\sum_{k\neq 0} \left| \int f(x)\phi_t(-x + k) \right| \le \left[C2^{n+\varepsilon}\|f\|_1 \sum_{k\neq 0} |k|^{-n-\varepsilon} \right] t^{\varepsilon} \xrightarrow{t\to 0} 0.$$

On the other hand, if we define $g = f\mathbb{1}_Q \in L^1(\mathbb{R}^n)$,

$$\lim_{t\to 0} \int_Q f(x)\phi_t(-x)\,dx = \lim_{t\to 0} g * \phi_t(0) = g(0) = f(0).$$

So we get that $f^t(0) \to f(0)$. $\square$